Arizona Mathematics Standards Revision – Expert Panel Review

Reviewer Name Marilyn Carlson

As you conduct your review of the **introduction**, please consider the following questions.

A. Does the introduction provide sufficient information and guidance on how to read the standards?
B. Does the introduction provide sufficient information on how the standards are structured?
C. Is there anything missing that should be included in the introduction?

1. **Please provide feedback on the introduction section. Include strengths as well as suggestions for refinements.**

The introduction serves its primary purpose of telling people how to read the standards and how they are structured. There is also a very well-written set of narratives describing the mathematical practice standards and excellent examples on fluency progressions, and I appreciate the emphasis on building procedural fluency from conceptual understanding. Perhaps the best part of the introduction about what the standards are intended to do compared to what they are not intended to do (such as outline specific teaching practices). The following are some specific comments related to the introduction.

1. On Pg. 12 you have "Key Considerations for Standards Implementation", which begins "There are important distinctions among different types of addition/subtraction and multiplication/division problems…" You then go on to have an excellent chart demonstrating how to implement a variety of problem types within a single domain to support flexible reasoning and robust understandings. However, it currently reads as if addition/subtraction and multiplication/division is the only area in which this applies and can be leveraged. I suggest a more broadly stated introduction that discusses the power of this approach throughout grade school mathematics and across topics and providing one or two additional examples from higher grade levels. Similarly, Table 3 and the text that precedes it are excellent examples of how to think about content across grades as a progression of fluency with related ideas and skills. I recommend making sure that people reading this understand that this is just one example of such a progression and is not THE fluency progression to focus on. I recommend a more broadly stated introduction and more examples to help make this point.

2. On page 18, you write "When formulas are presented within a specific grade level, students must be provided opportunities to gain conceptual understanding. _*The formula should be provided*_ (emphasis mine) and formula mastery should include conceptual understanding as well as use of the formula." To be consistent with your goal of having conceptual understanding at the foundation of procedural skill and fluency, I argue this should say "The formula should be developed from a foundation of conceptual understanding, and formula mastery should include this understanding as well as use of the formula in specific applied problems."

As you conduct your review of the **glossary**, please consider the following questions.

A. Does the glossary identify key terms and resources?
B. Do the definitions provide sufficient guidance for practitioners?
C. Is there anything missing that should be included in the glossary?

2. Please provide feedback on the glossary section. Include strengths as well as suggestions for refinements.

The basic structure of the glossary is fine, and most of the definitions are quite acceptable. The following notes are suggestions for changes, additions, or questions about the items included, many of which refer to transformations in Geometry, an area that is especially important to “get right” considering the standards are calling for some fundamental shifts to how key ideas are defined and leveraged in the course.

1. ”Quantities” and “quantitative reasoning” are mentioned numerous times throughout the standards, and these terms connect to a rich area of research into teaching and learning within mathematics education and have specific meanings and implications for teaching and learning. However, the terms are never truly defined in a satisfactory way (including examples) considering the important role they play in the standards and the coherence quantitative reasoning provides to grade school mathematics. In addition to expanding a discussion about the meanings of these terms elsewhere, they certainly deserve a place in the glossary. It seems that the definition of “quantitative reasoning” is taken as a given, but my research group’s extensive work with secondary teachers we find that few, if any, understand “quantitative reasoning” in a way that helps them develop materials and plan and implement instruction relative to this idea.

2. “Similarity transformation” is included but not “congruency transformation”. In addition, considering how the definition of congruence and similarity (defined relative to transformations) differs from many teachers’ background experiences, these terms should both be included in the glossary.

3. In “Rigid Motion” it should also be noted that such transformations map points to points, lines to lines, line segments to line segments, rays to rays, angles to angles, (etc.).” Understanding this property is key to leveraging transformations rigorously to prove theorems. For example, justifying the vertical angle theorems using transformations requires that one knows that a 180 degree rotation through any point on the line carries the line back onto itself. By giving a more complete definition of rigid motion it makes it more clear about what transformations would count as a rigid motions (and why others would not) as well as provides a more rigorous and mathematically sound foundation for using transformations in meaningful ways to prove theorems.

4. Reflection should be included (since all of the other transformations are included, and reflection in particular can benefit from a clear definition that describes how the image and preimage relate to the line of reflection.

5. In defining transformations, you sometimes say that the transformation impacts all points on a figure, other times that it affects all points on a coordinate system. Transformations should be generally considered a transformations of a plane (regardless of whether one has imposed a coordinate system on the plane) and this should be consistent across definitions of the transformations.

As you conduct your review of the **standards**, please consider the following questions.

A. Does each standard clearly state what students should know and be able to do?
B. Can the standards be measured?
C. Is there clarity in the standards? Are there any ambiguous or unclear words/phrases (some, a few, follow, understand…)?
D. Do the standards in each domain have sufficient **breadth of content or skill**?
E. Do the standards within a domain represent a range of **cognitive demand and rigor**?
F. Is there meaningful alignment and development of skills/knowledge allowing students to build understanding from one grade level to the next?
G. Are the standards written with clear student expectations that would be interpreted and implemented consistently across the state?

3. Please provide feedback on the Counting and Cardinality (CC) Domain (Kindergarten only). Include strengths as well as suggestions for refinements.

Consider having a standard related to grouping together objects in group sizes other than 10. With any size groups (including groups of 10), create groups and use the physical act of grouping to support the development of skip counting and foster a conceptual understanding of grouping that supports base ten reasoning. This could also be addressed under the NBT domain.

Other than that, the standards are clear and coherent and seem to be measureable and meaningful.

4. Please provide feedback on the Operations and Algebraic (OA) Thinking Domain (Grades K-5). Include strengths as well as suggestions for refinements.

This set of standards is clear and coherent with a solid and meaningful progression of ideas across grade levels.

5. Please provide feedback on the Number and Operations in Base Ten (NBT) Domain (Grades K-5). Include strengths as well as suggestions for refinements.

Consider having a standard related to grouping together objects in group sizes other than 10. With any size groups (including groups of 10), create groups and use the physical act of grouping to support the development of skip counting and foster a conceptual understanding of grouping that supports base ten reasoning. Asking students to create grouping schemes using a base other than 10 can help support reasoning about the base 10 system and highlight its benefits and historical/biological reasons why humans widely adopted this system. This could also be addressed under the CC domain as well.

K.NBT.B.2: "Demonstrate conceptual understanding of addition and subtraction through 10 using a variety of strategies." This does not meet the clarity criterion. If you want students to understand something "conceptually", be explicit about what meanings you want them to develop. "Conceptual understanding of addition and subtraction" is very vague.

6. **Please provide feedback on the Measurement and Data (MD) Domain (Grades K-5). Include strengths as well as suggestions for refinements.**

K.MD.A.1: "Describe several measureable attributes…" and K.MD.A.2: "Directly compare two objects with a measureable attribute in common..." Elsewhere in my feedback I mentioned how the terms "quantities" and "quantitative reasoning" are mentioned several times in the standards but are never defined and explained in any detailed way (which is very problematic since there is a rich body of research related to quantitative reasoning in mathematics education research). This standard is really the starting point for supporting quantitative reasoning, but it is not defined relative to the term "quantitative reasoning" and so any teacher seeking to understand what it means to engage in quantitative reasoning is not supported in seeing how these standards relate to that goal. This continues throughout this strand. You could rename the strand "Measurement, Data, and Quantitative Reasoning, or you could include a detailed description of what the standards writers mean by "quantitative reasoning".

3.MD.C.7: Part (d): "Understand area as additive by finding the areas of rectilinear figures." Since additive and multiplicative reasoning are well-defined concepts in mathematics education, and area calculations are multiplicative comparisons to a unit, the wording of this standard is problematic. My interpretation is that you want students to understand that they can break up plane figures, find the area of each part, and sum these areas to find the area of the original figure. If that is the case, consider rewording this standard to make this clearer as it was in the original standards. Perhaps something like "Understand that rectilinear figures can be decomposed into non-overlapping rectangles and that the sum of the areas of these rectangles is identical to the area of the original rectilinear figure." This is an understanding and is not really a "how to teach it" directive (you specifically talk about decomposition and composition skills with shapes in the Geometry standards, so this is clearly within the realm of reasoning skills you want students to develop and not a prescriptive teaching method – see 6.G.A for an excellent example of this that seems at odds with your motivation for changing part (d) of this standard). The current wording does not capture the mathematical idea that you intend.

7. **Please provide feedback on the Number and Operations-Fractions (NF) Domain (Grades 3-5). Include strengths as well as suggestions for refinements.**

3.NF.A.3: Part (a) "Understand two fractions as equivalent if they represent the same size part of the whole, or the same point on a number line." There are numerous ways to interpret fractions, and "part to whole" is only one way (so you are specifically emphasizing one interpretation), and in many research studies is a way of thinking that leads to students thinking that fractions have a value less than one (the part is smaller than the whole). I suggest rewording this standard, perhaps something like "Understand two fractions as equivalent if they have the same relative size compared to 1 whole." Essentially the wording should support interpretations that foster flexibility in applying the reasoning to numbers less than 1 and greater than 1.

4.NF.A.2 – This standard is fairly dense (and seems to contain multiple ideas that could be assessed independently). Consider writing it with subparts (a), (b), etc.

5.NF.B.3: "Interpret a fraction as division of the numerator by the denominator (a/b = a divided by b)…" This does not seem quite right to me. a/b is a number. It is the result of dividing a by b. a/b and a "divided by" b are not just two equivalent ways to write the same operation. They mean different things. a/b represents how many times as large a is compared to b, which is calculated by dividing a by b. We should be encouraging students to flexibly see fractions as numbers (a/b is a number that is a times as large as 1/b) as you have called for elsewhere, not as a command to calculate something that encourages them to see a/b as an a, and a bar, and a b, instead of seeing a/b as a number that could be interpreted as the <u>result</u> of a calculation.

You mention "1 whole" many times, but there doesn't appear to be a standard explicitly tied to reasoning about fractions related to a whole that is not thought of as "1" in some other unit. For example, if there is a bag of apples, students can visually represent (using number line reasoning or similar visualizations) how to interpret 4/5 of the bag of apples. If they are later told that the bag had 30 apples in it, then (4/5)(30) also represents 4/5 of "1 whole" but in units of "apples" now instead of "bags of apples". It's possible that this is already included, and maybe you intend for this reasoning to be supported in 5.NF.B.4, but this flexibility in understanding and moving between "1 whole" (that is, the value of a quantity using its own magnitude as the measurement unit" and the size of this whole (and any multiplicative comparisons to this whole) using other measurement units is extremely important and should be specifically highlighted and encouraged in the standards (and is a measureable standard).

8. **Please provide feedback on the Geometry (G) Domain (Grades K-8). Include strengths as well as suggestions for refinements.**

In the 8th grade standards you should expand the understanding of what is preserved under rigid transformations. Tt should also be noted that rigid transformations map points to points, lines to lines, line segments to line segments, rays to rays, angles to angles, (etc.).” Understanding this property is key to leveraging transformations rigorously to prove theorems (and not just to pay lip-service to the notion that transformations form the foundation of congruence and similarity). For example, justifying the vertical angle theorems using transformations requires that one knows that a 180 degree rotation through any point on the line carries the line back onto itself. By giving a more complete definition of rigid motion it makes it more clear about what transformations would count as a rigid motions (and why others would not) as well as provides a more rigorous and mathematically sound foundation for using transformations in meaningful ways to prove theorems.

Otherwise these standards are coherent and follow both a logical progression as well as being placed at grade levels to support understanding in other strands at those levels.

9. Please provide feedback on the Ratio and Proportion (RP) Domain (Grades 6-7). Include strengths as well as suggestions for refinements.

6.RP.A.1: “Understand the concept of a ratio…” What do you want them to understand? If you aren’t explicit, then you fail your question G: there are no clear expectations that will be broadly interpreted in the same way across schools. There is a lot of research about productive meanings for ratio in the literature, so there is no reason not to be explicit here.

10. Please provide feedback on the Number Systems (NS) Domain (Grades 6-8). Include strengths as well as suggestions for refinements.

These standards are coherent and logical.

11. Please provide feedback on the Expressions and Equations (EE) Domain (Grades 6-8). Include strengths as well as suggestions for refinements.

6.EE.B.6: “Use variables to represent numbers…” Students should be expected to see variable as a way of representing all of the values of a varying quantity, and see evaluating a function for a value of an input variable or solving an equation relative to choosing from among all of these possible values some subset that produce a given outcome. Math education research findings have repeatedly documented that students emerge from grade school mathematics without a strong concept of variation and tend to see variables as just unknowns, the one value that when substituted for *x* makes a statement true. We need to specifically support students in initially seeing variables as a letter that stands for the varying values of a varying quantity (varying distance in feet of a car from a stop sign as it drives away from the stop sign). Formulas and functions should then be introduced as constructs that define how two varying quantities are changing together (how they covary). Again, numerous researchers have documented that seeing variables as varying and functions as defining how two quantities change together are essential ways of thinking for understanding fundamental ideas in calculus. Variation and covariational reasoning should be supported from the earliest possible moments in students’ mathematical experiences. [This comment applies to the entirety of the EE strand]. After students have established a covariation view of functions the idea of a variable as an unknown can be logically introduced when “solving an equation for some value of the input quantity when a value of the output quantity is given” (e.g., give f(x) = 5 x – 9, solve 17 = 5x – 9 for x.)

In the high school standards (at least the current ones) it explicitly discusses evaluating as linked to using an input value to determine the corresponding output value, and solving an equation as using an output value to determine the corresponding input value. I argue that this way of thinking and terminology should be used in grades 6-8 as appropriate both because they are extremely powerful ways to think about the processes by also because it opens up multiple solution paths and methods for checking the reasonableness of solutions.

12. Please provide feedback on the Statistics and Probability (SP) Domain (Grades 6-8). Include strengths as well as suggestions for refinements.

7.SP.C.5: “Understand that the probability of a chance event is a number between 0 and 1 that expresses the likelihood of an event occurring.” Be more specific, since the wording of this standard allows for teachers to specifically teach, and students to develop, fuzzy and unproductive meanings for probability (for example a person saying “there’s a 50-50 probability because I don’t know what the outcome will be) that are not consistent with the mathematical definitions of probability. Replace this standard with the definition of probability as the long-term relative frequency of an event.

13. Please provide feedback on the Functions (F) Domain (Grades 8). Include strengths as well as suggestions for refinements.

I am concerned about the absence of covariational reasoning in the standards as a way of thinking about functions (representing the coordination of the values of two co-varying quantities such that a graph emerges as a trace showing constraints in how the quantities change in tandem, formulas as a representation of the restriction on how the values of varying quantities change together). Again, there is a broad body of research demonstrating the importance of covariational reasoning in developing a robust and powerful meaning for functions and representations of function relationships. For example, seeing graphs as emergent while coordinating the values of covarying quantities helps students avoid the common "trap" of seeing graphs as pictures of an event or physical objects (such as a wire). Having students work with dynamic visualizations of events and construct emergent graphical representations by tracking how the values of two quantities change together should be included and emphasized in the standards, not just because it helps students understand graphs in Algebra courses, but because it develops key insights that support a conceptual development of the ideas of Calculus.

14. Please provide feedback on the Algebra 1 (A1) standards. Include strengths as well as suggestions for refinements.

I really appreciate the fact that the standards have been broken down by course (A1, G, A2). This is by far the best change in the standards.

I am not as convinced about the benefits of stripping out the examples. In fact, I think the standards would benefit from multiple examples for EVERY standard. It seemed as if you found the inclusion of examples restricted the interpretation of the standard to only problem types like the given examples. I can understand that, but removing the examples creates a problem relative to your question G: "Are the standards written with clear student expectations that would be interpreted and implemented consistently across the state?" I am sure that the original purpose of including examples was to help ensure that the standards were interpreted in similar ways by all schools and by those creating the tests. Removing the examples makes it more likely that a variety of interpretations will exist (including those inconsistent with the intentions of the standards authors). Therefore, I recommend including several examples of questions where each standard would apply (at least 3) so that everyone reading the standard understands its purpose in similar ways but also sees the variety of ways in which the standard can be applied so that the examples do not create an overly narrow interpretation.

I am concerned about the absence of covariational reasoning in the standards as a way of thinking about graphs (representing the coordination of the values of two co-varying quantities such that a graph emerges as a trace showing constraints in how the quantities change in tandem), as well as functions in general. There is a wide body of research demonstrating the importance of covariational reasoning in developing a robust and powerful meaning for graphical representations and their connections to other representations (tables and formulas) and in avoiding the common "trap" where students see graphs as pictures of an event or physical objects (such as a wire). Having students work with dynamic visualizations of events and construct emergent graphical representations by tracking how the values of two quantities change together should be included in the standards, not just because it helps students understand graphs in Algebra courses, but because it develops key insights that support a conceptual development of the ideas of Calculus.

15. Please provide feedback on the Geometry (G) standards. Include strengths as well as suggestions for refinements.

I applaud the fact that you have retained a focus on transformations as the foundation of congruence and similarity in Geometry.

Similar to my comments for Algebra I and Algebra II, I believe that removing the examples was a mistake. I think the standards would benefit from multiple examples for EVERY standard. It seemed as if you found the inclusion of examples restricted the interpretation of the standard to only problem types like the given examples. I can understand that, but removing the examples creates a problem relative to your question G: "Are the standards written with clear student expectations that would be interpreted and implemented consistently across the state?" I am sure that the original purpose of including examples was to help ensure that the standards were interpreted in similar ways by all schools and by those creating the tests. Removing the examples makes it more likely that a variety of interpretations will exist (including those inconsistent with the intentions of the standards authors). Therefore, I recommend including several examples of questions where each standard would apply (at least 3) so that everyone reading the standard understands its purpose in similar ways but also sees the variety of ways in which the standard can be applied so that the examples do not create an overly narrow interpretation.

Moving the equations of conic sections to the plus standards was an excellent choice as it does not belong in Geometry, and is also not a necessary learning goal for standard Algebra 2 students.

G.G-CO.B.6: I made a comment about the definition of rigid motion for the glossary section, but I will repeat it here (because it needs to be beefed up), as well as make the case that the definition of rigid motion should be written out in the standard specifying exactly what students should learn about it. A rigid motion is a transformation that maps points to points, lines to lines, line segments to line segments with the same length (and thus preserves the distances between two points and their image points), rays to rays, and angles to angles of the same measure. The definition in the glossary only talks about preserving lengths and angle measures, but without the full definition you lose a lot of the rigor of proofs based on transformation arguments. For example, you can rigorously justify the vertical angle theorem using transformations only if you establish that a 180 degree rotation of a line using any point on the line as the center of rotation maps the line onto itself. Doing this for the two intersecting lines (using the intersection point as the center of rotation) ensures that the vertical angles map onto one another, which means that they have the same measure (since angle measure is preserved). It isn't quite enough to only say that lengths and angle measures are preserved for a rigorous proof.

16. Please provide feedback on the Algebra 2 (A2) standards. Include strengths as well as suggestions for refinements.

I really appreciate the fact that the standards have been broken down by course (A1, G, A2). This is a nice improvement in the standards.

I encourage the writers to reconsider using examples to make more the intent of each standard more clear. In fact, I think the standards would benefit from multiple examples for EVERY standard. It seemed as if you found the inclusion of examples restricted the interpretation of the standard to only problem types like the given examples. I can understand that, but removing the examples creates a problem relative to your question G: "Are the standards written with clear student expectations that would be interpreted and implemented consistently across the state?" I am sure that the original purpose of including examples was to help ensure that the standards were interpreted in similar ways by all schools and by those creating the tests. Removing the examples makes it more likely that a variety of interpretations will exist (including those inconsistent with the intentions of the standards authors). Therefore, I recommend including several examples of questions where each standard would apply (at least 3) so that everyone reading the standard understands its purpose in similar ways but also sees the variety of ways in which the standard can be applied so that the examples do not create an overly narrow interpretation.

In the introduction to the A2 standards you discuss the seeming importance of transformations and want students to draw generalizations about the graphs of all functions affected by the same kind of transformation [related to standard A2.F-BF.B.3]. I think we really miss the boat when we restrict our focus of transformations to graphical representations (and there is no indication in the standards that you intend the exploration to extend beyond graphical representations). Transformations of functions can be a rich area of exploration where a focus on the relative inputs and outputs of functions with a relationship like $g(x) = f(x - 2)$ can help students focus on the meaning of arguments, function outputs, domains and ranges, relationships of function values represented in tables, using a transformation to modify a formula if, say, you want to change the units of the input or output quantity, etc. This supports function reasoning, multiple representations, etc., connecting to countless other standards in the course, but almost none of this gets leveraged when the focus is only on graphical representations. In addition, a lot of the research into covariational reasoning demonstrates that students tend to think of graphs like pictures or static wires, and transformations as manipulations of some physical objects as opposed to an emergent model of how two quantities change together, and the research is pretty clear about the relatively dire implications for students with the former view. I highly recommend expanding and revising the transformations standards to explicitly go beyond graphical representations and to make connections to other standards that can be leveraged and supported with this broader scope. I also think that students' common misunderstandings about graphs, including ways of thinking about transformations, can be addressed by supporting covariational reasoning and including its development as a goal within the standards [I am out of space here – see my A1 comments.]

I applaud the move to space out the statistics standards. As it was, Algebra 2 was very bloated with standards and the set of statistics standards expected to be taught at that level was just too overwhelming. Moving some to Algebra I and some to plus standards and fourth year courses was a good move. Ideally, I would have liked Arizona to follow CCSS initial recommendations to include most of probability in Geometry instead of the algebra courses (and that is still my first choice and something I think you should consider). <See next two pages for my summary statements.>

17. Please provide any additional comments about this draft that you want the revision committee to consider.

I am surprised that there are not more standards explicitly calling on technology applications in the classroom. You state the essential role that technology plays in teaching and learning mathematics, and the fact that students who use mathematics outside of the classroom will be leveraging technology in doing so. Yet without explicitly calling for technology use in the standards beyond what most teachers interpret as using graphing calculators (such as having students writing code for algorithmic processes, students learning program iterative processes into spreadsheet software, etc.) you are missing perhaps the most vital ways that technology can both engage students and help them apply their learning outside of the classroom. The initial AzCCRS standards, and these revised standards, represent great strides in moving mathematics learning into the 21st century, but they can and should go further. Writing algorithmic procedures in code can be done with most graphing calculations that have been around since the mid-1990s, and spreadsheet software is many decades old, yet we still have not managed to integrate even these tools into our learning goals for students even though they would benefit our students.

I also think that we can do our teachers a great service by partnering with businesses and business leaders to create a repository of REAL applied problems from real career fields to support our goals of using modeling effectively to teach mathematics. We seem to take as given the importance of using “real life problems” (modeling with mathematics) in teaching, and I do believe this is true (especially when it comes to developing an engaging curriculum). However, most of the time these “application problems” are at best canned scenarios that little resemble the way that the mathematics is used by professionals in various fields. It’s one of the largest challenges we face – how to facilitate transfer of mathematical learning into other fields and into students’ real lives – and I believe we are missing a huge opportunity by not helping to facilitate this. I know that the committee doesn’t want the standards to become a curriculum, but we are not taking our roles seriously enough if we are not also working to provide support in the areas the standards are meant to inform (content, pedagogy, etc.) as a related effort.

I also question the benefits of stripping out examples from the standards at all levels. In fact, I think the standards would benefit from multiple examples for EVERY standard. It seemed as if you found the inclusion of examples restricted the interpretation of the standard to only problem types like the given examples. I can understand that, but removing the examples creates a problem relative to your question G: “Are the standards written with clear student expectations that would be interpreted and implemented consistently across the state?” I am sure that the original purpose of including examples was to help ensure that the standards were interpreted in similar ways by all schools and by those creating the tests. Removing the examples makes it more likely that a variety of interpretations will exist (including those inconsistent with the intentions of the standards authors). Therefore, I recommend including several examples of questions where each standard would apply (at least 3) so that everyone reading the standard understands its purpose in similar ways but also sees the variety of ways in which the standard can be applied so that the examples do not create an overly narrow interpretation.